MAT260 - Numerical Analysis - Cheat Sheet

Tommy Odland, June 6, 2016

Preliminaries

Trig $\sin(u \pm v) = \sin u \cos v \pm \cos u \sin v$

Norms $\|AB\| \leq \|A\| \cdot \|B\|$
$\|A + B\| \leq \|A\| + \|B\|$

Euler's method and beyond

FWE: $\mathbf{y}_{n+1} = \mathbf{y}_n + h\mathbf{f}_n$, explicit, not A-stable, first order

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Trapezoidal: $\mathbf{y}_{n+1} = \mathbf{y}_n + \frac{h}{2}\left[\mathbf{f}_n + \mathbf{f}_{n+1}\right]$, implicit, A-stable, second order

2. order AB: $\mathbf{y}_{n+2} = \mathbf{y}_{n+1} + \frac{h}{2}\left(-\mathbf{f}_n + 3\mathbf{f}_{n+1}\right)$ explicit, not A-stable, second order

Multistep methods

$$\sum_{m=0}^{s} a_m \mathbf{y}_{n+m} = h \sum_{m=0}^{s} b_m \mathbf{f}(t_{n+m}, \mathbf{y}_{n+m}) \tag{1}$$

Stability Stable if the *root criterion* is satisfied for LHS of (1):
(i) All zeros must reside in the closed unit disc, (ii) and all zeros of unit modulus must be simple(not repeated).

Order To obtain order p it is necessary and sufficient that, for $k \geq 1$:

$$\sum_{m=0}^{s} a_m = 0$$

$$\sum_{m=0}^{s} m^k a_m = k \sum_{m=0}^{s} m^{k-1} b_m$$

$$\sum_{m=0}^{s} m^{p+1} a_m \neq (p+1) \sum_{m=0}^{s} m^p b_m$$

where $0^0 := 1$.

A-stability The highest order of an A-stable multistep method is 2.
No explicit multistep method is A-stable.

Runge-Kutta methods

Polynomial fit If we wish to approximate $f(x)$ by the polynomial $P_n(x) = \sum_{k=0}^{n} a_k \varphi_k(x)$, where $\varphi_i(x)$ and $\varphi_j(x)$ are orthogonal with respect to the weight function $w(x)$, such that:

$$\langle \varphi_i(x), \varphi_j(x) \rangle = \int_a^b \varphi_i(x) w(x) \varphi_j(x)\; dx = 0$$

then the minimum is obtained by setting:

$$a_k = \frac{\langle \varphi_k(x), f(x) \rangle}{\langle \varphi_k(x), \varphi_k(x) \rangle}$$

IRK An implicit Runge-Kutta scheme is given by:

$$\xi_j = \mathbf{y}_n + h \sum_{i=1}^{\nu} a_{j,i} \mathbf{f}(t_n + c_i h, \xi_i)$$

$$\mathbf{y}_{n+1} = \mathbf{y}_n + h \sum_{j=1}^{\nu} b_j \mathbf{f}(t_n + c_j h, \xi_j)$$

RK tableux The RK-tableux is:

$$\begin{array}{c|c} \mathbf{c} & A \\ \hline & \mathbf{b}^T \end{array}$$

Stiff equations

Stability domain Assuming $\mathbf{y}_{n+1} = r(z)\mathbf{y}_n$, the linear stability domain is:

$$\mathcal{D} = \{z \in \mathbb{C} : |r(z)| < 1\} \tag{2}$$

Stability domain A method is A-stable if it preserves the decay of $\mathbf{y}' = \lambda \mathbf{y}$ when $\operatorname{Re}\lambda < 0$:

$$\mathbb{C}^- := \{z \in \mathbb{C} : \operatorname{Re} z < 0\} \subseteq \mathcal{D}$$

where $\mathcal{D}$ is defined in equation (2).

Error control

PECE The predictor-corrector scheme is:
(*i*) use *explicit* method to predict $\mathbf{y}_{n+1}$, then (*ii*) use *implicit* method to correct the initial prediction.

Error control If we integrate from $0 < t < T$ with step size h, and we wish that the total error is below ε, then the single step error must be below $\varepsilon h/T$. Here it is assumed that errors do not accumulate.

Nonlinear algebraic equations

Fix-point theorem The iteration $\mathbf{u}^k = \mathbf{g}(\mathbf{u}^{k-1})$ is a contraction mapping if:

$$\left\|\mathbf{g}(\mathbf{u}^k) - \mathbf{g}(\mathbf{u}^{k-1})\right\| < \gamma \left\|\mathbf{u}^k - \mathbf{u}^{k-1}\right\|$$

for some real $\gamma < 1$.

Convergence The fix point iteration $\mathbf{u}^{k+1} = \mathbf{g}(\mathbf{u}^k)$ converges if $|\mathbf{g}'(\mathbf{u}^k)| < 1$ for all relevant $\mathbf{u}$.

A bound

$$\|\mathbf{g}(\mathbf{v}^n) - \mathbf{g}(\mathbf{u}^n)\| < \frac{\gamma^n}{1-\gamma} \left\|\mathbf{v}^0 - \mathbf{u}^0\right\|$$

Newtons method Newtons method is:

$$\mathbf{x}_1 = \mathbf{x}_0 - J^{-1}(\mathbf{x}_0)F(\mathbf{x}_0)$$

Finite difference schemes

Gershgorin Every eigenvalue of A lies within at least one of the Gershgorin discs $D(a_{ii}, R_i)$, where a_{ii} is a diagonal element and $R_i = \sum_{j \neq i} |a_{ij}|$.

SOCD Second order central difference approximation:

$$u''(x) \approx \frac{1}{h^2}\left[u(x+h) - 2u(x) + u(x-h)\right] + \mathcal{O}(h^2)$$

For more information, please see:
(1) Iserles, Arieh. *Numerical Analysis of Differential Equations*. 2nd edition. Cambridge ; New York: Cambridge University Press, 2008.
(2) Richard L. Burden. Numerical Analysis. 3rd ed. Prindle, Weber & Schmidt, 1985.