MAT252 cheat sheet (continuum mechanics)

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Spectral theorem

Summation $b_i = \sum_j a_{ij} x_j = a_{ij} x_j \qquad \operatorname{div}(\mathbf{v}) = v_{i,i} = \frac{\partial v_i}{\partial x_i} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + ...$

Symmetry The matrix M is symmetric if $M^T = M$. If $M^T = -M$,the matrix is antisymmetric. Every matrix M can be expressed as:

$$M = A + S = \underbrace{\frac{1}{2}\left(M - M^T\right)}_{\text{antisymmetric}} + \underbrace{\frac{1}{2}\left(M + M^T\right)}_{\text{symmetric}}$$

Gram-Schmidt Given vectors $\{u_1, u_2, ..., u_n\}$, do:

$u_1 := u_1/||u_1||$
$u_2 := u_2 - (u_2 \cdot u_1)u_1$
$u_2 := u_2/||u_2||$
$u_3 := u_3 - (u_3 \cdot u_1)u_1 - (u_3 \cdot u_2)u_2$
$u_3 := u_3/||u_3||$
$u_n := u_n - \sum_i (u_n \cdot u_i)u_i$

To create a set in $\mathbb{R}^n$, start with u_1 and require that u_i be orthogonal to all previous by solving system with $(i-1)$ unknowns.

Navier-Stokes equations in polar coordinates

Gradient The gradient is $\operatorname{grad} \mathbf{v} = \frac{\partial \mathbf{v}}{\partial x_j} \otimes \mathbf{e}_j = \frac{\partial (v_i \mathbf{e}_i)}{\partial x_j} \otimes \mathbf{e}_j$

Divergence The divergence is $\operatorname{div} \mathbf{v} = \frac{\partial \mathbf{v}}{\partial x_j} \cdot \mathbf{e}_j = \frac{\partial (v_i \mathbf{e}_i)}{\partial x_j} \cdot \mathbf{e}_j$.
Digergence of tensor $\operatorname{div} \mathbf{T} = \frac{\partial \mathbf{T}}{\partial x_k} \cdot \mathbf{e}_k = \frac{\partial}{\partial x_k}\left[T_{ij}\left(\mathbf{e}_i \otimes \mathbf{e}_j\right)\right] \cdot \mathbf{e}_k$

Tensor rule $(a \otimes b)c = a(b \cdot c)$

Navier-Stokes $\rho \frac{D\mathbf{u}}{Dt} = \nabla \cdot \boldsymbol{\sigma} + \rho \mathbf{f} \Leftrightarrow m\mathbf{a} = \sum_i F_i$

Euler $\rho \frac{D\mathbf{u}}{Dt} = \nabla \cdot -p\mathbf{I}$ (No viscosity)

Material derivative $\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla$

Polar coordinates and solid-body rotation

Solid body rotation The solid body rotation is $v(r,\theta) = (v_r, v_\theta) = (0, \omega r)$ in polar coordinates and $v(x,y) = (v_x, v_y) = \omega(-y, x)$ in Cartesian

Lagrangian variables in two dimensions

Strain rate tensor Let $\mathbf{v}$ be a velocity field (deformation mapping), the strain rate tensor is: $d_{ij} = \frac{1}{2}\left(d_{i,j} + d_{j,i}\right)$

Change of v

$$\nabla \mathbf{v} = \underbrace{\mathbf{d}}_{\text{strain rate}} + \underbrace{\mathbf{w}}_{\text{rotation}} = \underbrace{\mathbf{d}_f}_{\text{pure strain}} + \underbrace{\mathbf{d}_e}_{\text{expansion}} + \underbrace{\mathbf{w}}_{\text{rotation}}$$

where $\mathbf{d}_e = \frac{1}{3} \operatorname{tr} \mathbf{d}$ and $\mathbf{d}_f = (\mathbf{d} - \mathbf{d}_e)$.

Ideal gas computations

Ideal gas law The ideal gas law is $PV = nRT$, where n is moles and R is a gas constant.

First law of thermodynamics The 1st law is $dU = dq - dw$, where U is the internal energy of the fluid, q is heat into fluid and w is work onto environment.

Entropy Entropy S is given by $dS = \frac{dq}{T}$

Exact differential dQ is exact $\Leftrightarrow$ Q is a state function $\Leftrightarrow \oint dQ = 0$

Second law of thermodynamics

Carnot engine (1) Isothermal expansion, (2) Adiabatic expansion, (3) Isothermal compression, (4) Adiabatic expansion. *Isothermal* means constant temperature, *adiabatic* means no heat transfer.

Efficiency of CE Efficiency is given by $\eta = \frac{W_{out}}{Q_{in}} = 1 - \frac{T_L}{T_H}$

Carnot theorem (1) Between T_H and T_L, no engine beats the efficiency of the CE.
(2) Efficiency η of CE is only a function of T_H and T_L.

Clausius postulate Heat never flows from hot to cold without aid(work input).

One-dimensional models for an elastic beam

Strain ℓ is the length of the bar, $u(x)$ is displacement

$$\varepsilon = \frac{\Delta \ell}{\ell_0} = \frac{\mathrm{d}u}{\mathrm{d}x}$$

Hookes law σ is stress, ε is strain, E is modulus of elasticity

$$\sigma = E\varepsilon$$

Wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\rho}{E}\frac{\partial^2 u}{\partial t^2}$$

Stress and displacement in a beam with longitudinal loading

Strain sdf

For general math, see [3]. For fluids, see [2]. For thermodynamics, see [4]. For solid mechanics, see [1].

References

[1] Lubliner, J., and Papadopoulos, P. *Introduction to Solid Mechanics: An Integrated Approach.* Springer New York: New York, NY, 2014.

[2] Pijush K. Kundu. *Fluid mechanics*, 5th ed. ed. Academic Press Elsevier, 2012.

[3] Tadmor, E. B., Miller, R. E., and Elliott, R. S. *Continuum Mechanics and Thermodynamics: From Fundamental Concepts to Governing Equations.* Cambridge University Press, Cambridge, 2011.

[4] Walter J. Moore. *Physical chemistry*, 3rd ed. ed. Prentice-Hall chemistry series. Prentice-Hall, 1962.