Cheat Sheet MAT251 *Tommy Odland – Mechanics – Edited: November 30, 2016*

▷ General mechanics

Degrees of freedom minimum number of independent generalized coordinates needed to describe system.

Variables

	Translation	Rotation
Momentum	$P = mv$	$L = r \times mv = I\dot{\omega}$
Δ Mom.	$F = ma$	$N = r \times F$
Energy	$T = \frac{1}{2}mv^2$	$T = \frac{1}{2}I\dot{\omega}^2$

Inertia w.r.t axis If r is the perpendicular distance from x, then

$$I_x = \sum_{m_i \in \mathcal{B}} r^2 m_i = \int_{\mathcal{B}} r^2 dm$$

Parallel axis thm If we know I_x, and r is the distance between I_x and $I_{x'}$, then:

$$I_{x'} = I_x + mr^2$$

Conserved quantities (1) Energy, (2) linear momentum and (3) angular momentum.

▷ Lagrangian mechanics

Hamilton's principle The true evolution of a system is the path that minimizes

$$I = \int_a^b L\left(q(t), \dot{q}(t), t\right) \, dt$$

and I is minimized when $\delta I = 0$. A necessary condition is that the EL eqns are satisfied.

The Lagrangian The Lagrangian is given by:

$$L = T - V$$

where T is kinetic energy and V is potential energy. It is useful to note that:

$$T_{\text{total}} = T_{\text{translation}} + T_{\text{rotation}}$$

Cyclic variables A variable q is cyclic if $\partial L / \partial q = 0$. In other words $L \neq L(q)$ and q is a conserved quantity. Typical in systems with rotational invariance.

Euler-Lagrange eqns

$$\left(\frac{\partial L}{\partial q_i}\right) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0 \qquad , \forall \text{ variables } q_i$$

Lagrange multiplier method Instead of reducing number of generalized coordinates, use the constraint $f(q_i, t) = C$ to set up

$$L \mapsto L' = L + \lambda \underbrace{\left(f(q_i, t) - C\right)}_{\text{constraint}}$$

The constraint force is $F_i = \frac{\partial \lambda f(q_i,t)}{\partial q_i}$

▷ Hamiltonian mechanics

Canonical variables Let $H(q,p)$ be the Hamiltonian, then

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} \qquad \dot{q}_i = \frac{\partial H}{\partial p_i} \tag{1}$$

(q, p) are the canonical variables, and any transformation preserving eqn (1) is canonical.

Hamiltonian from Lagrangian Compute canonical momenta

$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

and use $H = \sum_i \dot{q}_i p_i - L$, eliminate $\dot{q}_i$ from equation so that $H = H(q,p)$.

Legrende transform The transformation $L(q, \dot{q}) \mapsto H(q,p)$ is a Legrende transform. For $f(x)$, the Legrende transform is:

$$f^*(x^*) = \sup_x \left(x^* x - f(x)\right)$$

▷ Numerical methods

Symplecticity A method $y^{i+1} = \Phi_h\left(y^i\right)$ is symplectic iff $J = MJM^T$, where J is the canonical symplectic matrix $\left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$ and $M = \partial \Phi_h\left(y^i\right) / \partial y^i$.

Canonical transformation A change of variables $(q,p) \mapsto (Q,P)$ is symplectic if $J = MJM^T$, where $M = \partial(\text{new})/\partial(\text{old}) = \partial Q_i / \partial q_i$. A trick if the variable change is difficult is to express the equation as $AM = B$ and transform to $M = A^{-1}B$.

$$H\left(q\left(Q,P\right), p\left(Q,P\right)\right) = K\left(Q,P\right)$$

▷ Numerical methods cont.

Linearization Assume that we have $\dot{\mathbf{x}} = f(\mathbf{x})$. To examine around a stationary point $\mathbf{x}_0$, write $\mathbf{x} = \mathbf{x}_0 + \Delta\mathbf{x}$. Then $\dot{\mathbf{x}} = f(\mathbf{x}_0 + \Delta\mathbf{x}) \approx f(\mathbf{x}_0)$ and

$$\dot{\mathbf{x}} = \mathbf{J}(\mathbf{x}_0)\Delta\mathbf{x} = \mathbf{J}(\mathbf{x}_0)\Delta\mathbf{x}$$

Examine the eigenvalues and eigenvectors of $\mathbf{J}(\mathbf{x}_0)\Delta$ to see the behavior of the system.

Discrete Euler-Lagrange eqns We minimize

$$\sum_{i=1}^{M-1} \left[\partial_1 L\left(q^i, q^{i+1}\right) + \partial_2 L\left(q^{i-1}, q^i\right)\right] \delta q^i$$

by requiring that

$$\partial_1 L\left(q^i, q^{i+1}\right) + \partial_2 L\left(q^{i-1}, q^i\right) = 0$$

Construct discrete Lagrangian L_D like so

$$L_D = h\left[T\left(\frac{q_{i+1} - q_i}{h}\right) - V\left(q_i\right)\right]$$

An important check When solving problems related to numerics, remember to check behavior as $h \to 0$. It should look like the continuous case.