Cheat Sheet MAT230

Non-linear differential equations – Tommy Odland – Last edit: June 6, 2017

GENERAL NOTES

▷Preliminaries

- The general framework for ODEs is

$$\dot{\mathbf{x}} = f(\mathbf{x}), \quad \mathbf{x}(0) = \mathbf{x}_0.$$

  - Time is just a variable x_i, and higher-order derivatives can be reduced to first order derivatives by introducing new variables.
  - A well posed problem has a unique solution with continuous dependence on the initial conditions.
  - If f is continuous and all its partial derivatives are continuous, a unique solution exists.
- A function satisfies a **Lipschitz condition** if there exists an A such that $|f(x_2) - f(x_1)| \leq A|x_2 - x_1|$. If $f'(x)$ is limited, such an A exists. If this condition is satisfied, a unique solution exists.
  - Continuous derivative implies Lipschitz, but not vice-versa.
- Since explicit solutions are difficult to produce and hard to interpret, the common procedure of analysis is instead the following:

$$\text{System} \to \text{Trajectories} \to \text{Solution info}$$

- Flows on the line are governed by $\dot{x} = f(x)$, i.e. one dimensional maps.
- A **fixed point** $\mathbf{x}^*$ is a point such that $f(\mathbf{x}^*) = 0$.
- The **logistics equation** is given by $\dot{N} = rN(1 - N/K)$, where $r > 0$ is a growth rate and K is a carrying limit. The solution is $N(t) = K\left(1 + e^{-kt}\right)^{-1}$, and the behavior can be deduced from looking at the fixed points $f(N^*) = 0$.
- The idea of **linear stability analysis** is to linearize around a fixed point x^*. Introducing $\eta(t) = x(t) - x^*$ implies $\dot{x} = \dot{\eta}$, and we obtain

$$\dot{\eta} = \dot{x} = f(x^* + \eta) \approx \eta f'(x^*),$$

  and $f'(x^*)$ (or $F'(x^*)$) determines the linearized behavior around x^*.
- **Potentials** can be used to infer behavior from the system. The potential $V(x)$ is defined as

$$f(x) = -\frac{dV}{dx},$$

  and we have

$$V(x^*) = \min \quad \Rightarrow \quad x^* \text{ is stable,}$$
$$V(x^*) = \max \quad \Rightarrow \quad x^* \text{ is unstable.}$$

- **Picard iteration**
  Given $\dot{x} = f(x)$ and $x(t_0) = x_0$, we estimate the solution in the vicinity of x_0 by a sequence of approximations:

$$\begin{aligned} x_0 &= x_0 \\ x_1 &= x_0 + \int_{t_0}^{t} f(x_0)dt \\ x_2 &= x_0 + \int_{t_0}^{t} f(x_1)dt \\ \vdots &= \vdots \\ x_n &= x_0 + \int_{t_0}^{t} f(x_{n-1})dt \end{aligned}$$

▷Bifurcations

- A **bifurcation** is a qualitative change in behavior.
- Normal forms of the most common bifurcations:
  - **Saddle node**
    $$\dot{x} = r + x^2$$
  - **Pitchfork**
    $$\dot{x} = rx - x^3$$
  - **Transcritical**
    $$\dot{x} = rx - x^2$$
  - **Subcritical pitchfork**
    $$\dot{x} = rx + x^3$$
- The purpose of **dimensional analysis** is to write an equation in dimensionless form. Two advantages are (1) "small" can be defined as $\ll 1$ and (2) the number of parameters are reduced by introducing dimensionless groups.
  - How to non-dimensionalize: (1) set a new time scale $\tau = t/T$ and a new dependent variable $N = x/a$, (2) find the derivatives with respect to τ, (3) make every term non-dimensional and (4) put the unknowns into non-dimensional groups.

▷Linear systems

- A **linear system** is of the form

$$\dot{\mathbf{x}} = A\mathbf{x}, \quad A \in \mathbb{R}^{2\times 2}.$$

- The eigenvalues of A determine the behavior in the phase plane, since the general solution is

$$\mathbf{x}(t) = C_1 e^{\lambda_1 t}\mathbf{v}_1 + C_2 e^{\lambda_2 t}\mathbf{v}_2$$

- In general: nodes, saddles and spirals can occur.

- A rough sketch is:[1]
  - real(λ) > 0 implies growth.
  - real(λ) < 0 implies decay .
  - imag(λ) $\neq 0$ implies a spiral/center.
  - If $\lambda_1 = \lambda_2$ we have a star or degenerate node, depending on the number of eigenvectors. If $\lambda_i = 0$ we have a non-isolated fixed point.

▷Phase plane ($\mathbb{R}^2$)

- A good website for plotting phase planes is `http://comp.uark.edu/~aeb019/pplane.html`.
- Quantitative behavior with exact formulas is often unattainable, so we settle for qualitative behavior.
- **Nullclines** are curves where either $\dot{x} = 0$ or $\dot{y} = 0$.
- Linearizing around $f(\mathbf{x}) = 0$ gives correct information unless $\text{Re}(\lambda_i) = 0$ for some i in the Jacobian matrix of the linearization. If $\text{Re}(\lambda_i) = 0$ the point is "fragile"—linearizing does not always give the correct answer.
- The **basin of attraction** for a point $\mathbf{x}^*$ is the subset of the phase plane which sends all trajectories to $\mathbf{x}^*$.
- **Stability**
  - **Poincare stability** (stability of path)
    Let $\mathcal{H}^*$ be the half path for the solution $x^*(t)$. Then $\mathcal{H}^*$ is Poincare stable iff for every $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ such that
    $$||x(t_0) - x^*(t_0)|| < \delta \Rightarrow \max_x \text{dist}(\mathcal{H}^*, \mathcal{H}) < \epsilon$$
    where $x(t)$ is a neighboring solution with path $\mathcal{H}$.
  - **Liapunov stability** (stability of solution)
    The solution $x^*(t)$ is Liapunov stable for $t \geq t_0$ iff for every $\epsilon > 0$ there exists a $\delta(t, \epsilon) > 0$ such that
    $$||x(t_0) - x^*(t_0)|| < \delta \Rightarrow ||x(t) - x^*(t)|| < \epsilon$$
    for all $t \geq t_0$, where $x(t)$ is any neighboring solution.
  - **Uniform stability** (stability for all t)
    If a solution $x^*(t)$ is stable and δ if the definition of Liapunov stability is independent of t_0 then $x^*(t)$ is uniformly stable. I.e. the path is always close to neighboring paths for every t_0.
  - **Asymptotic stability** (stability in the limit)
    If $x^*(t)$ is a stable solution, and in addition there exists a $\delta(t_0) > 0$ such that
    $$||x(t_0) - x^*(t_0)|| < \delta \Rightarrow \lim_{t\to\infty} ||x(t) - x^*(t)|| = 0$$
    then $x^*(t)$ is asymptotically stable. $x(t)$ is a neighboring solution.
- A **conserved quantity** is a real valued, non-zero, continuous function $E(\mathbf{x})$ such that $\dot{E}(\mathbf{x}) = 0$. The quantity $E(\mathbf{x})$ is conserved along trajectories.
  - A conservative system cannot have any attracting fixed points.
  - If a point $\mathbf{x}$ is a local minimum for $E(\mathbf{x})$, then $\mathbf{x}$ is a (perhaps non-linear) center.
- A **Hamiltonian system** has
  $$\dot{x} = \frac{\partial H}{\partial y} \quad \dot{y} = -\frac{\partial H}{\partial x},$$
  where $H(x, y)$ is the Hamiltonian. $H(x, y)$ is constant along solution paths, since $\dot{H}(x, y) = 0$.
- A system is **reversible** if it's invariant under a change of variables $\mathbf{x} \mapsto R(\mathbf{x})$, $t \mapsto -t$.
  - One common mapping is $y \mapsto -y$, $t \mapsto t$. (Reflection over x-axis.)
- The **index of a curve** C is how many counterclockwise rotations the vector $f(\mathbf{x})$ makes as $\mathbf{x}$ moves in a counterclockwise closed trajectory around C once.
  $$I_C = \frac{1}{2\pi}[\phi]_C$$
  Colloquially, I_C measures the "winding" of $f(\mathbf{x})$ along the curve C.
  - As long as a continuous deformation $C \mapsto C'$ does not cross through a fixed point, the index is the same, i.e. $I_C = I_{C'}$.
  - If C does not enclose fixed points, $I_C = 0$.
  - For a saddle, $I_{\mathbf{x}} = -1$. For other types of fixed points $I_{\mathbf{x}} = +1$.
  - If I_C encloses fixed points $\mathbf{x}_1^*, \mathbf{x}_2^*, \ldots, \mathbf{x}_n^*$ then
    $$I_C = \sum_{i=1}^n I_i$$

▷Asymptotics

- **Big O** and **little o**
  - $f(x) = O(g(x))$ if $\lim_{x\to x_0} f(x)/g(x) < \infty$, in terms of order $f(x) \leq g(x)$ as $x \to x_0$.
  - $f(x) = o(g(x))$ if $\lim_{x\to x_0} f(x)/g(x) = 0$, in terms of order $f(x) < g(x)$ as $x \to x_0$.
- An **asymptotic sequence** of functions $\{\phi_n\}$ has $\phi_{n+1}(x) = o(\phi_n(x))$ as $x \to x_0$.
- An **asymptotic series** is of the form
  $$f(x; \epsilon) \sim \sum_{k=0}^n a_k(x)\phi_k(\epsilon), \quad \epsilon \to \epsilon_0.$$
  The coefficients are calculated using
  $$a_n(x) = \lim_{\epsilon\to\epsilon_0} \frac{f(x; \epsilon) - \sum_{k=0}^{n-1} a_k(x)\phi_k(\epsilon)}{\phi_n(x)}.$$

[1] A dynamic website on linear systems: `http://mathlets.org/mathlets/linear-phase-portraits-matrix-entry/`

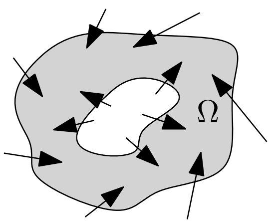
- An asymptotic expansion **does not have to converge** to be useful: as $n \to \infty$ for a fixed ϵ it does not have to converge for all x, but for a fixed N it should converge as $\epsilon \to \epsilon_0$ for all x.
- A sequence becomes **non-uniform** when ϕ_n does not dominate ϕ_{n+1}, i.e. when two adjacent terms are of the same order. The region of non-uniformity for $1 + \epsilon x + \epsilon^2 x^2 + \dots$ is $x = O(1/\epsilon)$.
- **Non-uniformity** arises from:
  - Infinite domains, due to secular terms.
  - ϵ in front of highest derivative.
- **Optimal truncation**: truncate in front of smallest term of the asymptotic expansion.

▷Limit cycles(closed orbits)

- **Ruling out closed orbits**
  - If $\dot{\mathbf{x}} = -\nabla V$ (**gradient system**) there are no closed orbits.
  - If there exists a **Liapunov function** there is no closed orbit. A Liapunov function has:
    i. $V(\mathbf{x}) > 0$ for all $\mathbf{x} \neq \mathbf{x}^*$(pos.def).
    ii. $\dot{V}(\mathbf{x}) < 0$ for all $\mathbf{x}$ (downhill flow).
    And $V(\mathbf{x})$ must be continuously differentiable, i.e. continuous and continuous first derivatives.
  - **Dulac's criterion** states that if $\nabla \cdot (g\dot{\mathbf{x}})$ has one sign in Ω there are no closed orbits in Ω, where g is a scalar function. This is from the div. thm:

$$\iint \nabla \cdot \mathbf{F} \; dA = \oint \mathbf{F} \cdot \hat{\mathbf{n}} \; dr$$

    Some choices for g are $1, 1/(x^a y^b)$, e^{ax} and e^{by}.
  - **Bendixson's negative criterion** is Dulac's criterion with $g = 1$.
- **Finding closed orbits**
  - The **Poincaré-Bendixson theorem** states that if one can construct a trapping region then Ω must have a limit cycle if Ω has no fixed points.



- **Relaxation oscillations** operate on two time scales: a slow buildup and a fast release.
- The general weakly non-linear equation is

$$\ddot{x} + \omega_0^2 x = \epsilon F(x, \dot{x}),$$

  and a specific example is the Duffing equation $\ddot{x} + x + \epsilon x^3 = 0$.
- **Regular perturbation theory** consists of writing the solution as

$$x(t, \epsilon) = x_0(t) + \epsilon x_1(t) + \epsilon^2 x_2(t) + O(\epsilon^3).$$

▷Boundary layers

- Boundary layers arise when ϵ is in front of the highest order derivative, this causes a singularity.

▷Discrete maps and fractals

- A fixed point $x^* = f(x^*)$ has the following stability properties:
  - $|f'(x^*)| < 1 \Rightarrow$ Stable
  - $|f'(x^*)| > 1 \Rightarrow$ Unstable
  - $|f'(x^*)| = 1 \Rightarrow$ Linear analysis inconclusive
- Cobwebs can be used to picture iterated maps.
- The logistics map $x_{n+1} = rx_n(1 - x_n)$ exhibits chaos when $r \approx 3.5699$.
- The **similarity dimension** of an object/fractal is

$$D = \frac{\ln(m)}{\ln(r)},$$

  where r is the shrinking factor and m is the number of boxes needed to cover the object.
- The **box dimension** is

$$D = \lim_{\epsilon \to 0} \frac{\ln(N(\epsilon))}{\ln(1/\epsilon)},$$

  where $N(\epsilon)$ is the number of boxes of side length ϵ needed to cover the object.
  - The Cantor set has dimension $D = \ln(2)/\ln(3)$.

▷Chaos, attractors and Liapunov

- **Chaos** is aperiodic behavior in a deterministic system with sensitive dependence on the initial conditions.
- An **attractor** is a closed set A with the following properties:
  (a) All trajectories starting in A stay in A.
  (b) A attracts and open set of initial conditions.
  (c) A is minimal with respect to these properties.
- Assume that we have two solution trajectories $x(t)$ and $x(t) + \delta(t)$, then $||\delta(x)|| \sim ||\delta_0||e^{\lambda t}$, where λ is the **Liapunov exponent**. If $\lambda > 1$ there is sensitive dependence of the initial conditions.

RECIPES (TECHNIQUES)

▷Recipe: Strained coordinates

When to use: When straight forward Poincare expansion fails due to non-uniformities. Does not work if amplitude needs to be adjusted.

1. Introduce $\tau = t(1 + \epsilon w_1 + \epsilon^2 w_2 + \dots)$
2. Compute derivatives.
3. Insert into equation, collect terms.
4. Use freedom in w_1 to cancel terms giving rise to secular terms, i.e. terms that grow boundlessly as $t \to \infty$.

▷Recipe: Multiscale

When to use: When straight forward Poincare expansion fails due to non-uniformities. Has the power to change amplitude and frequency.

1. Construct a fast time $T_0 = t$ and a slow time $T_1 = \epsilon t$. Functions of the slow time T_1 will be regarded as constant on the fast time scale T_0. We have
$$x(t, \epsilon) = x_0(T_0, T_1) + \epsilon x_1(T_0, T_1) + O(\epsilon^2)$$
and the time derivative becomes
$$\dot{x} = \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\partial x}{\partial T_0} + \epsilon \frac{\partial x}{\partial T_1}.$$
2. Plug the series expansion into the equation, collect powers of ϵ. A typical solution for $O(1)$ is $A(T_1)e^{iT_0} + A^*(T_1)e^{-iT_0}$.
3. Plug into $O(\epsilon)$, get differential equation for $A(T_1)$ by setting resonant terms to zero.
4. Use $A(T_1) := R(T_1)e^{i\theta(T_1)}$ to get differential equations for $R(T_1)$ and $\theta(T_1)$.
5. Use initial conditions and solve.

▷Recipe: Averaging

When to use: in weakly non-linear oscillators, exemplified by $\ddot{u} + \omega_0^2 u = \epsilon F(u, \dot{u})$.

1. The unperturbed problem has solution $u(t) = a\cos(\omega_0 t + \theta)$. Assume the solution is $u(t) = a(t)\cos(\omega_0 t + \theta(t))$, where ω_0 is found in the unperturbed problem.
2. Assume that $\dot{u}(t) = -\omega_0 a \sin(\omega_0 t + \theta)$. Compare with $\dot{u}(t)_{\text{true}}$ to obtain one constraint. Compute $\frac{d}{dt}\left(\dot{u}(t)_{\text{ass}}\right)$ and plug into the problem to obtain second constraint.
3. Obtain differential equations for $a(t)$ and $\theta(t)$ from the above constraints.
4. Averaging the RHS of the differential equations give the averaging method.

▷Recipe: Boundary layers

When to use: Boundary value problems with ϵ in front of highest derivative.

1. Determine location of the boundary layer, determined by $a(x)$ in
$$\epsilon f'' + a(x)f' + b(x)f = c(x), \quad x_1 < x < x_2.$$
   (a) If $a(x) > 0$, then the BL is at $x = x_1$.
   (b) If $a(x) < 0$, then the BL is at $x = x_2$.
   (c) If $a(x)$ changes sign, then the BL is at $a(x) = 0$.
2. Choose $s = \pm(x - x_i)/\epsilon^p$ near the BL. Choose p so as to keep as many terms as possible.
3. Use the **Prandtl matching condition** (here with BL at x_1)
$$\lim_{s\to\infty} f_0^{\text{inner}} = \lim_{x\to x_0} f_0^{\text{outer}}$$
to determine coefficients, the limit is also f_0^{match}.
4. The solution becomes
$$f \sim f_0^{\text{inner}} + f_0^{\text{outer}} - f_0^{\text{match}}.$$

▷References

- Lecture notes.
- Steven H. Strogatz (Steven Henry). *Nonlinear Dynamics and Chaos: With Applications to Physics, Biology, Chemistry, and Engineering.* Second edition. Westview Press, a member of the Perseus Books Group, 2015.
- Alan W. Bush. *Perturbation Methods for Engineers and Scientists.* CRC Press Library of Engineering Mathematics. Boca Raton, Fla: CRC Press, 1992.